Physics 211C: Solid State Physics

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Topic 1: Fermi Liquid Theory

Basic idea (from notes): Adiabatic continuity relates interacting problem to non-interacting one.

Steps:- ⓐ • Start from non-interacting $\sum_k \epsilon_k c_k^+ c_k$

ⓑ • Slowly turn on interactions ("slow" quantified later)

ⓒ • If the ⓑ can be done at all, the new ground state is

$$|\Psi_t\rangle = T e^{-i\int_{-\infty}^{0} H_I(t)\,dt} \underbrace{|\Psi_{-\infty}\rangle}_{|FS\rangle}$$

where

$$\mathcal{H} = \underset{\substack{\uparrow \\ \text{bare}}}{H_0} + H_I\,(t=0) \equiv H_0 + H_I$$

Landau FLT states that if the above procedure can be carried out without encountering any phase transitions, then the low lying eigenstates $\left(\Delta E \sim \frac{1}{L^{\#}}\right)$ of $\mathcal{H}$ are 1-1 correspondence with eigenstates of H_0.

Precisely, if $|\Psi_{t=-\infty}\rangle = \prod_{k<k_F} c_k^+ |0\rangle$

$$\Downarrow$$

$$|\Psi_{t=0}\rangle = \prod_{k<k_F} a_k^+ |0\rangle$$

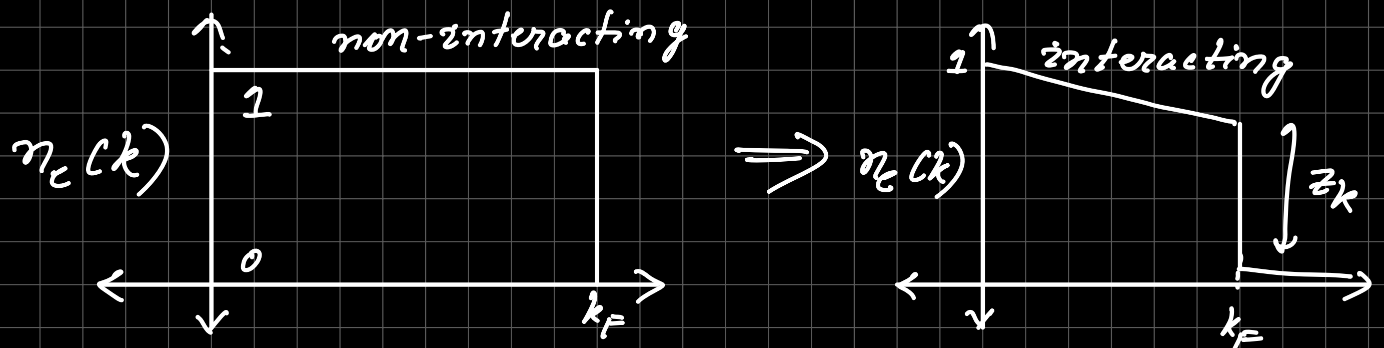
where $a_k^+ = U c_k^+ U^+$ is the QP operator

FLT assumes that a_k^+ & c_k^+ are related as

$$c_{k\sigma}^+ = \sqrt{Z}\, a_{k\sigma}^+ + A(\{k\})\, a_{k\sigma}^+ a_{k',\sigma'}^+ a_{k,\sigma'} + \cdots\cdots$$

where crucially $\sqrt{Z_k} \neq 0$ (can be taken to be almost the def^n of FLT)

Correspondingly, $n_k = \langle c_{k\sigma}^+ c_{k\sigma} \rangle$ has a discontinuity of Z_k at FS



Much like free $e^{\ominus}$ gas, analogous p-h excitations exist in the interacting system. These are the so called quasiparticles, possessing renormalized energies.

$$\xi = \frac{k^2}{2m} - \mu \longrightarrow \xi' = \frac{k^2}{2m^*} - \mu \quad \text{or} \quad m \to m^*$$

Although low en. eigenstates of H are in correspondence with H_0, not all excitations of H correspond to H_0

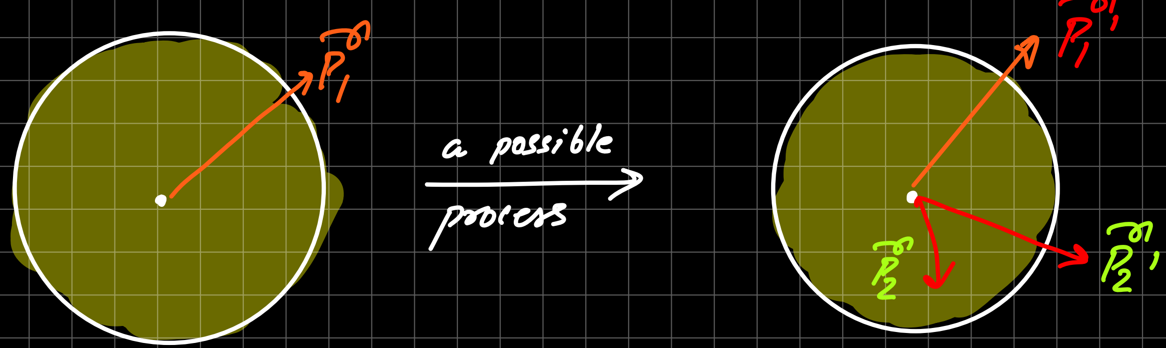
e.g. sound modes (perturbation of eigenstates)

$\Downarrow$

new excitations with no non-interacting counterpart

Scattering processes of Quasiparticles

- For a quantitative description of FLT & to account for m^*, ξ^* etc., we attempt to create an effective theory of interaction b/w QPs. This demands understanding scattering of QPs at low energies.
- Consider a single QP located outside FS, with momentum $\vec{p}$. For it to scatter, it must conserve charge, momentum & energy. Clearly, the only way this happens is by the following



momentum-energy conservation leads to

$$\vec{P}_1 = \vec{P}_1' + \vec{P}_2' - \vec{P}_2 \qquad (\text{-ve sign for holes})$$

$$P_1^2 = P_1'^2 + P_2'^2 - P_2^2$$

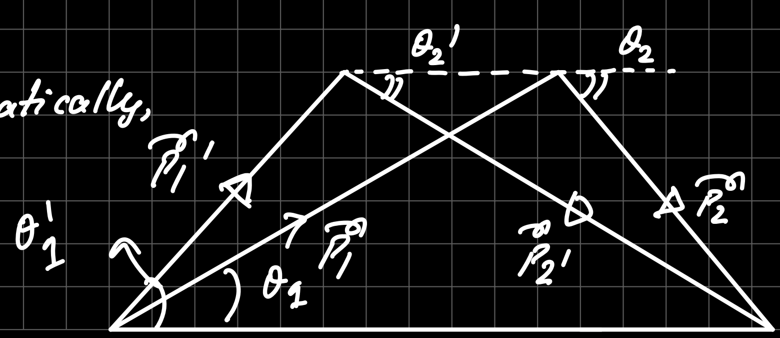
moreover $\{P_1, P_1', P_2'\} > P_F$ while $P_2 < P_F$

At $T=0$, all excitations are close to the FS, so assume $|P_1 - P_F| <<< P_F$

↓

sets other p_i near FS.

Schematically,



$$\vec{P}_1 + \vec{P}_2 = \vec{P}_1' + \vec{P}_2'$$

$\vec{p}$ conservat^n leads to

$$p_1 \cos\theta_1 + p_2 \cos\theta_2 = p_1' \cos\theta_1' + p_2' \cos\theta_2'$$

$\because$ all vectors are almost equal, at leading order we have

$$\theta_1 \equiv \theta_1' \equiv \theta_2 \equiv \theta_2'$$

(see that this ensures energy conservation too)

$$\Rightarrow p_1 + p_2 \approx p_1' + p_2'$$

From fermi's golden rule, scattering rate for such processes are given by

$$\Gamma \sim \int_{\vec{p_2}, \vec{p_1}'} \delta(p_1^2 + p_2^2 - p_1'^2 - p_2'^2)$$

($\because$ $\vec{p_2}'$ is completely determined by constraints)

$\because$ en. cons. is accounted for by $p_1 + p_2 \approx p_1' + p_2'$ (L's are almost equal)

$$(\text{angular integral}) \times \int dp_1' \, dp_2$$

subject $p_1 + p_2 = p_1' + p_2'$

now $p_2' > p_F \Rightarrow p_1 + p_2 - p_1' > p_F \Rightarrow p_1' < p_1 + p_2 - p_F$

$\because$ $p_1' > p_F \Rightarrow p_1 + p_2 > 2p_F \Rightarrow p_2 > 2p_F - p_1$

$\because$ $p_2 < p_F \Rightarrow 2p_F - p_1 < p_2 < p_F$

$$\therefore \Gamma \sim \int_{2p_F - p_1}^{p_F} dp_2 \int_{p_F}^{p_1 + p_2 - p_F} dp_1' \sim \frac{(p_1 - p_F)^2}{2}$$

$\therefore$ a QP with en. $\sim v_F(p_1 - p_F)$ has $\tau \propto \frac{1}{(en)^2}$ @ $T=0$.

$\because \frac{\Gamma}{e} \to 0$ as $e \to 0$, QPs remain well defined.

This also justifies adiabatic preparation

$\Downarrow$

① Time req. to prep an eigenstate containing a few QPs must be larger than inverse level spacing (i.e. $(\Delta \xi)^{-1} \propto L$) for adiabaticity to hold (recall Berry phase derivation)

(this refers to the previously alluded to "slow" turning of interaction)

$\therefore t_{prep} > L$

② Moreover, the time of preparatⁿ <<< Lifetime of QP (else QP will disintegrate while in prep)

now, $e \sim \frac{1}{L}$ while $\tau \propto \frac{1}{\xi^2} \propto L^2$

$\because O(L^2) >>> L$, one has sufficient time for preparing the QPs adiabatically.

Single Particle Green's function in a Fermi Liquid

$$G(t,k) = -i \langle g.s. | T c_k(t) c_k^+(0) | g.s. \rangle$$

$$= -i \,\Theta(t) \langle gs | c_k(t) c_k^+(0) | gs \rangle$$

$$+ i \,\Theta(-t) \langle gs | c_k^+(0) \, c_k(t) | gs \rangle$$

$$c_k(t) = e^{-it\tilde{\xi}_k} c_k(0) \qquad \tilde{\xi}_k = \xi_k = E_k - \mu$$

$$\langle gs | c_k c_k^+ | gs \rangle = 1 - n_F \qquad \text{(for free fermions)}$$

$$\therefore \; G(t,k) = \underbrace{-i\,\Theta(t)\, e^{-i\xi_k t} (1-n_F)}_{\Downarrow} + i\,\Theta(-t)\, n_F\, e^{-i\xi_k t}$$

$$n_F = \Theta(\xi_k)$$

$$\Rightarrow G(t,k) = -i\,\Theta(t) e^{-i\xi_k (t)}\, \Theta(\xi_k) + i\,\Theta(-t)\, \Theta(-\xi_k)\, e^{-i\xi_k t}$$

$\downarrow$ f.t.

$$G(\omega,k) = \frac{1}{\omega - \tilde{\xi}_k + i\cdot\epsilon\cdot \underbrace{\text{sign}(\omega)}_{\text{sign of real part of } \omega}}$$

$$\therefore \; G^{-1}_{free}(k,\omega) = \omega - \tilde{\xi}_k + i\eta\, \text{sign}(\omega)$$

* All this was for non-interacting system. For interacting system,

$$G^{-1}_{int}(k,\omega) = \omega - \tilde{\xi}_k - \underbrace{\Sigma(k,\omega)}_{\text{self energy}}$$

e.g:- ① $\Sigma(k,\omega) = \Delta\tilde{\xi}_k \Rightarrow$ new FS

$\tilde{\xi}_k + \Delta\tilde{\xi}_k = 0$

② $$\frac{1}{\omega - \tilde{\xi}_k + \frac{i}{\tau}} \xrightarrow{f.T.} \underbrace{e^{-t/\tau}}$$

Quasi-particle lifetime.

τ = life-time of q.p.

$\Sigma(k,\omega)$ is a model dependent quantity. It has to be calculated in a perturbative way.

(TG:- Landau theory $\Leftrightarrow$

hubbard model, large N, hubbard model in momentum space

integrable "1d" models

Bethe Ansatz solvable $\Rightarrow$

* integrability isn't well defined. $\Rightarrow$ still modern)

Expand G^{-1} near FS

$\Sigma = \mathrm{Re}\,\Sigma + i\,\mathrm{Im}\,\Sigma$

$$G^{-1} \approx \omega - \omega\left(\frac{\partial \mathrm{Re}\,\Sigma}{\partial\omega}\right)_{\omega=0}$$

$$- (k-k_F)\, \partial_k \left(\xi_k + \mathrm{Re}\, (\Sigma(k,0) \right) \Big|_{k=k_F}$$

$$- i\, \mathrm{Im} \left(\Sigma(k,\omega) \right)$$

$$Z^{-1} = 1 - \frac{\partial}{\partial \omega} \left(\mathrm{Re}\, \Sigma(k_F,\omega) \right)$$

$$\Downarrow$$

$$G \sim \frac{1}{\omega Z^{-1} + 1} = \frac{Z}{\omega - (\quad)}$$

$$\therefore G = \frac{Z}{\omega - \xi'_{renorm} - \dfrac{i}{\tau(k,\omega)}}$$

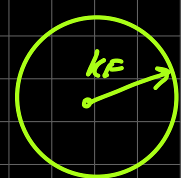
$$\xi_{renorm}(k) = Z(k-k_F)\, \partial_k \{ \xi_k + \mathrm{Re}\, \Sigma(k,0) \} \Big|_{k=k_F}$$

$$\Big\downarrow \text{ assume spherical FS}$$

$$= \frac{k^2 - k_F^2}{2m^*}$$

→ why not k^*_F ?

$\Downarrow$



Luttinger theorem

⇒ Volume of FS remains unchanged.

$$\frac{1}{\tau(k,\omega)} = -Z\, \mathrm{Im}\, \Sigma(k,\omega)$$

* Let's say $\frac{\partial}{\partial k} \mathrm{Re}(\Sigma(k,0)) = 0$

then $\boxed{\frac{1}{m^*} \sim Z}$

Essence of FL: $Z \neq 0$

destroy FL: $m^* \to \infty$ → phase transitions of a FL (Ref:- Senthil, 2008 paper)

Z → like an order parameter of some kind

$$G(t,k) = \frac{\underbrace{Z_k}_{\propto\, f^n \text{ of direction}} \, e^{-i\omega t}\, d\omega}{\omega - \xi_k^{renorm} + \frac{i}{\tau(k,\omega)}}$$

Acc. to Landau argument

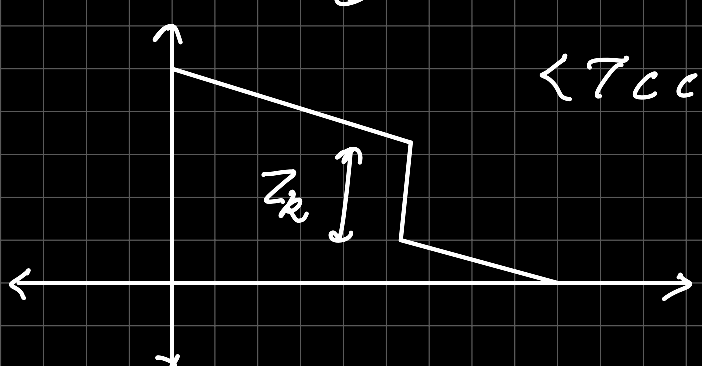
$$\tau(k) \sim \frac{1}{(k-k_F)^2}$$ → from phase space arguments

$$G(t,k) = \frac{\underbrace{Z_k}_{\propto\, f^n \text{ of direction}} \, e^{-i\omega t}\, d\omega}{\omega - \xi_k^{renorm} + \frac{i}{\tau(k,\omega)}}$$

$\frac{i}{\tau(k,\omega)} \to (k-k_F)^2$

$\xi_k^{renorm} = Z(k-k_F)\,\partial_k \xi \; - - - \}$

$$\therefore G(t, k \approx k_F) \approx Z_k \Big[-i\,\theta(t)\,\theta(\xi_k^{renorm})\, e^{-it\tilde{\xi}_k} + i\,\theta(-t)\,\theta(-\xi_k^{*})\, e^{-it\tilde{\xi}_{k_F}} \Big]$$



$\langle T c c^\dagger \rangle \sim \langle c_k^\dagger c_k \rangle$

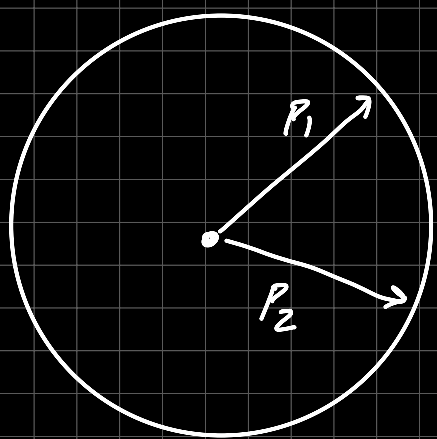
$t = 0^-$

Z_k → can be thought of as

$\underbrace{c^\dagger}_{\text{actual QP creator}} \sim \sqrt{Z}\ \underbrace{f^\dagger}_{\text{bare fermion}} + \underbrace{\cdots}_{\text{stuff we don't know}}$

$Z \to$ overlap b/w bare & QP

$\langle c^\dagger c \rangle \sim \underset{\sim}{Z} \langle f^\dagger f \rangle$



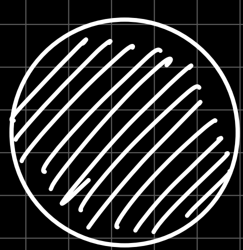
$P_1 - P_2 \longrightarrow P_1', P_2'$

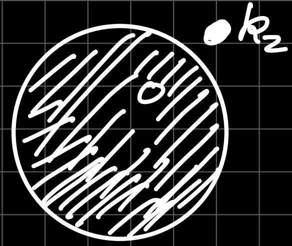
$$\left.\begin{aligned} \mathcal{H}_{Int} &\sim c_{k_1}^\dagger c_{k_2}^\dagger c_{k_1} c_{k_2} \\ &\sim n_{k_1} n_{k_2} \end{aligned}\right\}$$ very loose way of thinking it

Given distribution of q.p. $n_{p\sigma}$, define

even at finite temp. $\rightarrow$ $\delta n_{p\sigma} = n_{p\sigma} - \underset{\sim}{n_{p\sigma}(T=0)}$ = deviation from g.s. occupation

= tells us about # of excitations.





$\delta n_k = -1$

$\delta n_k = +1$

[Q. when does a FS break up into smaller puddles?]

Landau $\rightarrow$ says that $\delta n_{p\sigma}$ are the actual dof

$\rightarrow$ can write down $F(\delta n_{p\sigma})$ & then minimize it.

$$F[\{n_{p\sigma}\}] = \underbrace{E[\{n_{p\sigma}\}]} - T\,\underbrace{S[\{n_{p\sigma}\}]}$$

$n_{p\sigma} \to 0$ or 1

&

can use Shannon entropy

Landau postulated a form of E, lacking information.

<u>non.-int.</u>

$$E = \sum_{p\sigma} (\varepsilon^0_{p\sigma} - \mu)\, n_{p\sigma}$$

$n_{p\sigma} = 0,1$ (@ $T=0$)

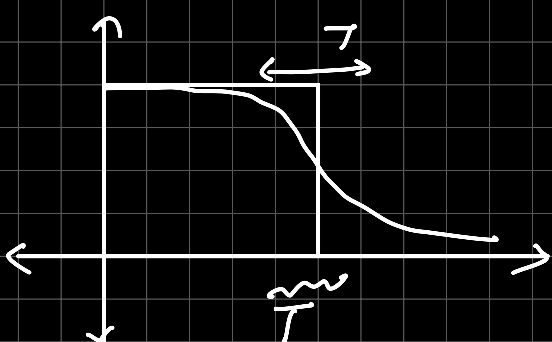
$$\varepsilon^0_{p\sigma} = \frac{p^2}{2m}$$

* Perhaps, then, for interacting case,

$$\left.\epsilon^0_{p\sigma} \to \epsilon_{p\sigma} = \frac{p^2}{2m^*}\right\}$$ but actually this is incorrect.

<u>Why?</u>

Our Guess for $E - E(T=0) = \sum_{p\sigma} (\underbrace{\overset{\approx T}{\epsilon_{p\sigma} - \mu}}_{\frac{p^2}{2m^*}})\ \underbrace{\overset{\approx T}{\delta n_{p\sigma}}}_{\frac{T}{\epsilon_F}}$



$$+ \frac{1}{2} \sum_{\substack{p\sigma \\ p'\sigma'}} \overset{\approx T}{\delta n_{p\sigma}}\ \overset{\approx T}{\delta n_{p'\sigma'}}\ f_{pp'\sigma\sigma'}$$

Landau

we need to keep every term of $O(T^2)$

(coleman has incorrect definition of $\delta n_{p\sigma}$)

(both terms are of the same order)

$$S[\{n_{p\sigma}\}] = -\left[\sum_{p\sigma} (n_{p\sigma} \log n_{p\sigma}) + (1 - n_{p\sigma}) \log (1 - n_{p\sigma})\right]$$

$\therefore$ now get $n_{p\sigma}$

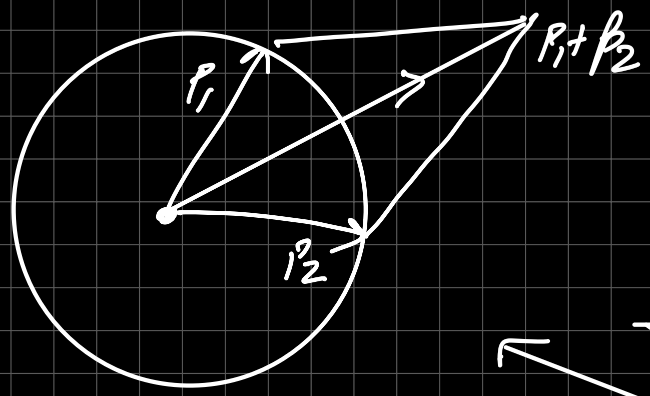
$$\frac{\delta F}{\delta n_{p\sigma}} = 0 \Rightarrow \xi_{p\sigma} + \underbrace{\sum_{p'\sigma'} f_{p\sigma\, p'\sigma'}\, \delta n_{p'\sigma'}}_{} = 0$$

energy of a single particle depends on distribution of all other particles

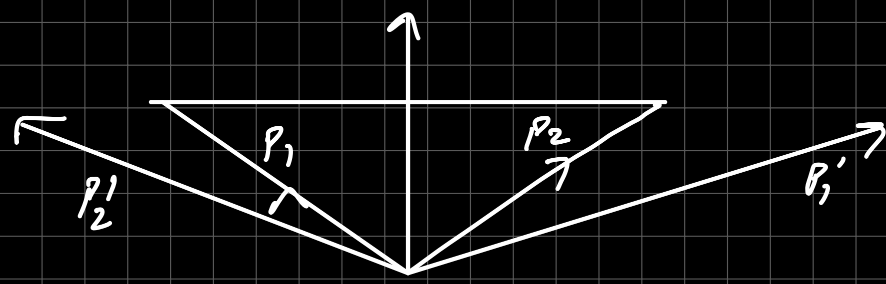
$$n_{p\sigma} = \frac{1}{e^{\beta(\xi_{p\sigma} - \mu)} + 1}$$ a self consistent equation.

$\xi_{p\sigma}(n_{p\sigma})$

essence of QP $\Rightarrow$ reduce # of parameters



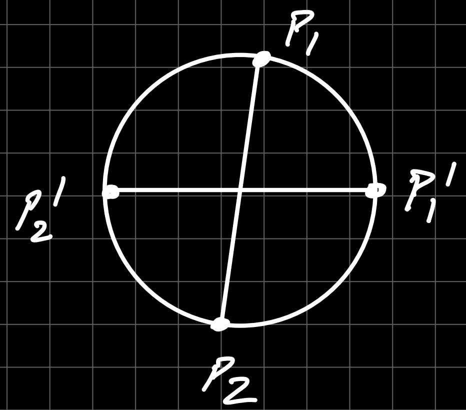
nested FS ↳ some flat portion ↗ in square lattice



$(p_1, p_2) \longrightarrow \underbrace{(p_1', p_2')}_{\text{continuum of solutions}}$ $\Big\} \Rightarrow \exists$ an instability to something

e.g.:- BCS

$p_1' = -p_2'$



$\underbrace{p_1 = -p_2}$ ↳ large # of solutions

& consequence

$$\chi_{sc}^{(\omega=0)} = \infty$$

$$\langle c^\dagger c^\dagger c c \rangle_{\text{fermions}} \sim \log(\omega)$$

for $\omega \neq 0$

↙ similarly

$$\int \langle \eta(x,t)\, \eta(0,0) \rangle e^{-iqx} e^{i\omega t} \sim \log(\omega)$$

$\therefore$ Instability $\Rightarrow$ continuous family of solution

↓

due to some RG

nested fermi surface has some instability

↳ log divergences due to multiple solutions